ISRAEL JOURNAL OF MATHEMATICS **127** (2002), 61–80

THE DISTAL ORDER OF A MINIMAL FLOW

BY

JOSEPH AUSLANDER

Department of Mathematics, University of Maryland
College Park, MD, 20742, USA
e-mail: jna@math.umd.edu

AND

ELI GLASNER

Department of Mathematics, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
e-mail: glasner@math.tau.ac.il

ABSTRACT

The capturing operation, a kind of reverse orbit closure, is defined. In terms of this operation generalized proximal relations $P_\gamma(X)$ for a minimal flow (X,T) are defined. The distal order of (X,T) is then defined to be the least ordinal number γ for which the distal structure relation $S_d(X) = P_\gamma(X)$. It is shown that the distal order of a minimal flow is either finite or ω, the first infinite ordinal. Examples are given which show that minimal flows of any allowable distal order actually occur. An analogous theory is developed for the equicontinuous relation $S_{eq}(X)$.

§0. Introduction

In this paper, we obtain characterizations of the distal and equicontinuous structure relations in minimal flows. These arise by means of iterative processes, starting with the proximal and regionally proximal relations, respectively. To every minimal flow, we can assign ordinal numbers, the distal and equicontinuous orders of the flow. It is shown that the distal order of a minimal flow is either finite or ω, where ω is the first infinite ordinal. An important element of these

Received February 27, 2000

constructions is the capturing operation, which is a kind of reverse orbit closure. We show that the distal structure and equicontinuous structure relations are the smallest closed capturing sets containing, respectively, the proximal and regionally proximal relations. In the final section of the paper, we present examples which show that minimal flows of any allowable distal order actually occur.

We now review some basic dynamical notions and establish our notations. A flow (X, T) is a (right) topological action $(x, t) \mapsto xt$ of a topological group T on a compact Hausdorff space X. The flow (X, T) is minimal if $\overline{xT} = X$ for all $x \in X$. If (X, T) and (Y, T) are flows, a homomorphism is a continuous equivariant map π from X onto Y (thus $\pi(xt) = \pi(x)t$ for $x \in X$, $t \in T$). In this case Y is said to be a factor of X, and X an extension of Y. The relation $R_\pi = \{(x, x') : \pi(x) = \pi(x')\}$ is a closed T invariant equivalence relation. Conversely, every closed T invariant equivalence relation R defines a factor flow $Y = X/R$ and a homomorphism π: $(X, T) \to (Y, T)$ such that $R = R_\pi$.

We will make extensive use of the Galois theory of minimal flows, as developed by Ellis. Let (M, T) be the universal minimal flow. This is the (unique) minimal flow of which every minimal flow with acting group T is a factor. Let G be the automorphism group of (M, T). Then G has a compact, T_1 topology, for which multiplication is separately continuous, and for which inversion is continuous. This topology is in general not Hausdorff, nor is multiplication jointly continuous.

The topology of G can be described in terms of graphs. For $\varphi \in G$, the graph of φ is the set $gr(\varphi) = \{(m, \varphi(m)) : m \in M\}$. The automorphism $\psi \in G$ is in the closure of the subset Φ of G iff $gr(\psi) \subset \text{closure}\{(m, \phi(m)) : \phi \in \Phi\}$, where the latter closure is taken with respect to the product topology in $M \times M$. In terms of nets: let $\{\varphi_j\}$ be a net in G, then $\varphi_j \to \varphi$ if there is a net $\{m_j\}$ in M with $m_j \to m$ such that $\varphi_j(m_j) \to \varphi(m)$.

To every minimal flow (X, T) one can associate a closed subgroup $\mathcal{G}(X)$ of G (the " Ellis" group of (X, T)) as follows. Let π: $M \to X$ be a homomorphism, and let $\mathcal{G}(X) = \{\varphi \in G : \pi\varphi = \pi\}$. (A different homomorphism gives rise to a conjugate subgroup.) Two minimal flows have the same Ellis group if and only if they are " proximally equivalent" — that is, they have a common proximal extension. For further details, see [A] and [E].

If (X, T) is a flow, then x and y in X are said to be proximal if for any $\alpha \in \mathcal{U}$ (the unique uniform structure of the compact space X) there is a $t \in T$ such that $(xt, yt) \in \alpha$. We denote the proximal relation by P. The flow (X, T) is said to be distal if $P = \Delta$, where $\Delta = \{(x, x) : x \in X\}$ is the "diagonal" in $X \times X$ (there are no non-trivial proximal pairs).

The flow (X,T) is called equicontinuous if the collection of self maps of X defined by T is an equicontinuous family: that is, if for every $\alpha \in \mathcal{U}$ there is a $\beta \in \mathcal{U}$ such that if $(x,x') \in \beta$ then $(xt, x't) \in \alpha$ for all $t \in T$ (" $\beta T \subset \alpha$").

Equicontinuity can be characterized in terms of regional proximality. The points x and y are called regionally proximal if for any $\alpha \in \mathcal{U}$ there are x' and y' with $(x,x') \in \alpha$ $(y,y') \in \alpha$ and $t \in T$ such that $(x't, y't) \in \alpha$.

We denote the regionally proximal relation by Q. Clearly $P \subset Q$. It is easy to see that a flow is equicontinuous if and only if $Q = \Delta$.

Every flow (X,T) has a maximal distal and a maximal equicontinuous factor, (X_d,T) and (X_{eq},T) respectively. That is, (X_d,T) is distal, and every distal factor of (X,T) is a factor of (X_d,T). (X_{eq},T) has the corresponding property for equicontinuous factors. Thus there are closed T invariant equivalence relations S_d and S_{eq} such that $X/S_d = X_d$ and $X/S_{eq} = X_{eq}$. Since an equicontinuous flow is distal, we have $S_d(X) \subset S_{eq}(X)$.

It is immediate that $P \subset S_d$ and $Q \subset S_{eq}$. However, in general neither P nor Q is an equivalence relation. Even if P is an equivalence relation, it need not be closed. (Q is always closed, and in many important cases is an equivalence relation in minimal flows.)

The following classical theorem of Ellis and Gottschalk [EG] characterizes S_d and S_{eq} in terms of P and Q.

THEOREM: *Let (X,T) be a flow. Then S_d is the smallest closed T invariant equivalence relation containing P and S_{eq} is the smallest closed T invariant equivalence relation containing Q.*

ACKNOWLEDGEMENT: We began our work on this paper during the special year in ergodic theory at the Institute for Advanced Studies of The Hebrew University in Jerusalem, 1996–7. We would like to thank the Institute for its support.

§1. The capturing operation

Let (X,T) be a flow. We define an operation on subsets of X, the "capturing" operation. If $K \subset X$, the capturing set of K, $C(K) = \{x \in X : \overline{xT} \cap K \neq \emptyset\}$. We say that K has the capturing property or is a capturing set if $C(K) = K$.

The following proposition lists elementary properties of the capturing operation and capturing sets.

PROPOSITION 1.1: *Let (X,T) be a flow.*

(i) *C is an idempotent operation. That is, for any $K \subset X$, $C(C(K)) = C(K)$.*

(ii) *The union and intersection of a family of capturing sets is capturing.*

(iii) *If $K \subset X$, and $t \in T$, $Kt \subset C(K)$. (Hence a capturing set is T invariant.)*

(iv) *If φ is an automorphism of (X,T) and $K \subset X$ then $\varphi(C(K)) = C(\varphi(K))$.*

(v) *For any subset K of X, there is a smallest closed capturing set K^* containing K.*

(vi) *Suppose K is a closed invariant subset of X. Then $y \in C(K)$ if and only if y is proximal to some point of K.*

(vii) *Let (Y,T) be a flow, let $\pi\colon X \to Y$ be a homomorphism, and let $L \subset Y$. Then $C(\pi^{-1}(L)) = \pi^{-1}(C(L))$. (Hence if L is capturing, so is $\pi^{-1}(L)$.)*

(viii) *Let K be a closed invariant set and let $z \in C(K)$ be almost periodic. Then $z \in K$.*

(ix) *Let K be an invariant capturing set. Then K is closed if and only if K and $\overline{K}$ have the same almost periodic points.*

Proof: We omit the easy proofs of (i), (ii), (iii), (iv) and (vii).

Let $\mathcal{L}$ be the family of closed capturing sets containing K. ($X \in \mathcal{L}$ so $\mathcal{L} \neq \emptyset$.) Clearly $K^* = \cap\{L : L \in \mathcal{L}\}$ is the smallest closed capturing set containing K.

To prove (vi), suppose that y is proximal to some $x \in K$. Let I be a minimal right ideal in $E(X,T)$, the enveloping semigroup of (X,T), such that $xp = yp$ for all $p \in I$. Since $yp = xp \in K$, we have $y \in C(K)$. Conversely, suppose $y \in C(K)$. Since K is closed invariant, there is a minimal right ideal I in $E(X,T)$ such that $yI \subset K$. Let u be an idempotent in I. Then $yu \in K$ and yu is proximal to y.

Let z be an almost periodic point of $C(K)$. Let $p \in \beta T$ such that $zp \in K$. Since z is almost periodic, there is a $q \in \beta T$ such that $z = zpq \in Kq \subset K$. This proves (viii).

Finally, suppose K is invariant, capturing, and that K and $\overline{K}$ have the same almost periodic points. Let $z \in \overline{K}$. Then if u is a minimal idempotent, zu is an almost periodic point in $\overline{K}$. Hence $zu \in K$, and $z \in C(K) = K$, which proves (ix). ∎

Our main interest in the capturing operation is as it is applied to the product flow $(X \times X, T)$. (T acts on the product by acting on each coordinate: $(x,x')t = (xt, x't)$.) Note that $C(\Delta)$ is just P, the proximal relation.

LEMMA 1.2: *Let (X,T) be a flow and let R be a closed invariant subset of $X \times X$. Then $PR \subset C(R) \subset PRP$. If (X,T) is minimal, then $C(R) = PR$.*

Proof: Let $(x,y) \in PR$ so $(x,x') \in P$ and $(x',y) \in R$ for some $x' \in X$. Let $p \in E(X,T)$, the enveloping semigroup of (X,T), such that $xp = x'p$. Since

R is closed invariant, $(x'p, yp) \in R$ so $(xp, yp) \in R$ and $(x, y) \in C(R)$. If $(x, y) \in C(R)$, let I be a minimal right ideal in $E(X, T)$ such that $(xp, yp) \in R$ for $p \in I$. If u is an idempotent in I then $(x, xu) \in P$ and $(y, yu) \in P$, so $(x, y) \in PRP$. If (X, T) is minimal, there is an idempotent $u \in I$ such that $xu = x$, so $(x, y) \in PR$. ∎

THEOREM 1.3: *Let (X, T) be a flow. Then $S_d(X)$ and $S_{eq}(X)$ are capturing sets in $(X \times X, T)$.*

Proof: Since $P \subset S_d$, $C(S_d) \subset PS_dP = S_d$, so $C(S_d) = S_d$, and similarly $C(S_{eq}) = S_{eq}$. ∎

§2. The image of the distal structure relation under a homomorphism

Now we consider the distal and equicontinuous structure relations in minimal flows. (The interested reader can find in [V] a proof of the relative version of the following result.)

THEOREM 2.1: *Let (X, T) and (Y, T) be minimal flows and let $\pi: X \to Y$ be a homomorphism. Then $\pi(S_d(X)) = S_d(Y)$ and $\pi(S_{eq}(X)) = S_{eq}(Y)$.*

The proof of Theorem 2.1 depends on two lemmas.

LEMMA 2.2: *Let (X, T) and (Y, T) be minimal flows and let $\pi: X \to Y$ be a homomorphism. Let K be a closed invariant subset of $X \times X$. Then $\pi(C(K)) = C(\pi(K))$. In particular (taking $K = \Delta$), $\pi(P(X)) = P(Y)$.*

Proof: It's easy to see that $\pi(C(K)) \subset C(\pi(K))$. Let $(y, y') \in C(\pi(K))$. Then there is a minimal idempotent $v \in \beta T$ such that $(y, y')v \in \pi(K)$. Let $(x, x') \in K$ such that $(x, x')v = (x, x')$ and $\pi(x, x') = (y, y')$. Let w and w' be minimal idempotents in βT such that $yw = y$, $y'w' = y'$, $wv = v$, and $w'v = v$. (This can be achieved by choosing v, w, and w' in the same minimal right ideal of βT.) Then $\pi(xw, x'w') = (y, y')$. Since $(xw, x'w')v = (xv, x'v) = (x, x') \in K$, we have $(xw, x'w') \in C(K)$. ∎

The next lemma concerns regular minimal flows. (X, T) regular means that all almost periodic points of $(X \times X, T)$ are on graphs of flow automorphisms. That is, denoting the automorphism group of (X, T) by Γ, if (x, y) is almost periodic, there is a $\varphi \in \Gamma$ such that $y = \varphi(x)$.

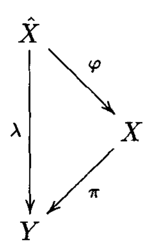
LEMMA 2.3: *Let (X,T) and (Y,T) be minimal flows with (X,T) regular, and let $\pi: X \to Y$ be a homomorphism. Let S be a closed T and Γ invariant equivalence relation on X and let $R = \pi(S)$. Suppose R is a capturing set. Then R is an equivalence relation.*

Proof: It is immediate that R is reflexive and symmetric. Since R is capturing, it is sufficient to show that if (y,y'), $(y',y'') \in R$ with (y,y',y'') an almost periodic point, then $(y,y'') \in R$. Let (x,x'), $(z',z'') \in S$ with $\pi(x,x') = (y,y')$ and $\pi(z',z'') = (y',y'')$. Without loss of generality, (x,x',z',z'') is an almost periodic point. Let $\varphi \in \Gamma$ with $\varphi(z') = x'$. Since $\pi(x') = \pi(z')$ we have $\pi\varphi = \pi$. Let $x'' = \varphi(z'')$. Then $\pi(x'') = \pi\varphi(z'') = \pi(z'') = y''$. Since S is Γ invariant, $(x,x'') \in S$ and $(y,y'') = \pi(x,x'') \in R$. ∎

Proof of Theorem 2.1: The proofs for the distal and equicontinuous structure relations are word for word the same, so we just give the proof for the former case.

Clearly $R = \pi(S_d(X)) \subset S_d(Y)$. For the other inclusion it suffices to show that R is an equivalence relation, since by Lemma 2.2 it contains $P(Y)$. Assuming first that X is regular, we observe that $S = S_d(X)$ is Γ-invariant and capturing (Theorem 1.3). Lemma 2.2 now implies that R is also capturing and we conclude by Lemma 2.3 that R is an equivalence relation, so that in this case $\pi(S_d(X)) = S_d(Y)$.

In the general case let $\hat{X}$ be any regular extension of X, e.g., the regularizer of X or M, the universal minimal flow. We now have the commutative diagram:



By the first part of the proof we now get

$$\pi(S_d(X)) \subset S_d(Y) = \lambda(S_d(\hat{X})) = \pi \circ \psi(S_d(\hat{X})) = \pi(S_d(X)). \qquad ∎$$

§3. A characterization of the distal structure relation for minimal flows

In order to characterize the distal structure relation for minimal flows, we define a family of "proximality" relations. Let $P_0 = \Delta$, the diagonal, $P_1 = C(P_0)$ (so $P_1 = P$). Inductively, if γ is an ordinal number and P_i has been defined for $i < \gamma$, define $P_\gamma = C(\overline{\bigcup_{i<\gamma} P_i})$. (Therefore $P_{\gamma+1} = C(\overline{P_\gamma})$.)

The following proposition follows easily from Proposition 1.1 and Theorem 1.3.

PROPOSITION 3.1: *Let (X,T) be a flow. Then, for every ordinal number γ, $P_\gamma \subset S_d$ and (if $\gamma > 0$) P_γ is a capturing set.*

Let $P^\#(X)$ be the smallest closed capturing set containing Δ. It follows from general principles that there is an ordinal number γ such that $P_\gamma(X) = P^\#(X)$. Since $S_d(X)$ is capturing we have $P^\#(X) \subset S_d(X)$. We show that for minimal flows, in fact $P^\#(X) = S_d(X)$. For this we require a lemma.

LEMMA 3.2: *Let (X,T) be a regular minimal flow with automorphism group Γ. Let K be a reflexive, symmetric, closed capturing set in $X \times X$. Suppose the set $F = \{\alpha \in G : gr(\alpha) \subset K\}$ is a subgroup of Γ. Then K is an equivalence relation.*

Proof: Suppose (x,y) and (y,z) are in K. Let u be a minimal idempotent in βT, and let $\bar{x} = xu$, $\bar{y} = yu$ and $\bar{z} = zu$. Then $(\bar{x},\bar{y})$ and $(\bar{y},\bar{z})$ are almost periodic points of K (by Proposition 1.1, K is invariant) so there are φ and ψ in Γ with $\bar{y} = \varphi(\bar{x})$ and $\bar{z} = \psi(\bar{y})$. Since $(\bar{x},\bar{y}) \in K$, $\varphi \in F$, and similarly $\psi \in F$. Thus $\bar{z} = \psi\varphi(\bar{x})$, and since F is a group, $\psi\varphi \in F$ and $(\bar{x},\bar{z}) \in K$. Finally, since K is capturing, it follows that $(x,z) \in K$. ∎

THEOREM 3.3: *Let (X,T) be a minimal flow. Then $S_d(X) = P^\#(X)$.*

Proof: By Theorem 1.3, $P^\# \subset S_d(X)$. For the other direction we observe that it is sufficient to show that $P^\#$ is an equivalence relation. In fact, since $P = C(\Delta)$ we have $P \subset P^\#$ and thus, $P^\#$ being closed and invariant (Proposition 1.1.(iii)), if it is also an equivalence relation we have $P^\# \supset S_d(X)$, as the latter is the smallest closed invariant equivalence relation containing P.

We first prove this for the universal minimal flow (M,T).

Let G be the automorphism group of (M,T). If $\varphi \in G$ it follows from Proposition 1.1 that $(id \times \varphi)(P^\#)$ is a closed capturing set.

Let $B = \{\varphi \in G : (id \times \varphi)(\Delta) \cap P^\# \neq \emptyset\}$. We show that B is a group. Note that B consists of those $\varphi \in G$ whose graphs meet $P^\#$ (equivalently, those $\varphi \in G$

whose graphs are contained in $P^{\#}$). Since $P^{\#}$ is symmetric, it follows that B is closed under inversion.

Let $\varphi \in B$. Since $\varphi^{-1} \in B$, $\Delta \subset (id \times \varphi)(P^{\#})$. By minimality of $P^{\#}$, we have $P^{\#} \subset (id \times \varphi)(P^{\#})$, and similarly $P^{\#} \subset (id \times \varphi^{-1})(P^{\#})$. Therefore $(id \times \varphi)(P^{\#}) = P^{\#}$ for all $\varphi \in B$, and it now follows easily that B is a group. It now follows from Lemma 3.2 that $P^{\#}$ is an equivalence relation in M.

Now, suppose (X, T) is a minimal flow, and let $\pi\colon M \to X$ be a homomorphism. Now, by Proposition 1.1, $\pi^{-1}(P^{\#}(X))$ is a closed capturing set in $M \times M$ containing $\Delta(M)$, so $P^{\#}(M) \subset \pi^{-1}(P^{\#}(X))$. Then $S_d(X) = \pi(S_d(M)) = \pi(P^{\#}(M)) \subset P^{\#}(X)$. Since $S_d(X)$ is capturing, we have $P^{\#}(X) \subset S_d(X)$, and the proof is completed. ∎

§4. The distal order of a minimal flow

Let (X, T) be a minimal flow. The **distal order** of (X, T) (denoted $\delta = \delta(X, T)$) is defined to be the least ordinal number γ for which the distal structure relation $S_d(X) = P_\gamma(X)$. ($\delta(X, T) = 0$ if and only if (X, T) is distal.) We will show that in fact for every minimal flow, $\delta(X, T) \leq \omega$ where ω is the first infinite ordinal. In the last section we will see that for every $k \leq \omega$ there exists a minimal flow (X, T) with $\delta(X, T) = k$.

LEMMA 4.1: *Let $\pi\colon X \to Y$ be a homomorphism of minimal flows. Let (y, y') be an almost periodic point in $\overline{P(Y)}$. Then there is an almost periodic point $(x, x') \in \overline{P(X)}$ such that $\pi(x, x') = (y, y')$.*

Proof: Let $(x_0, x_0') \in \overline{P(X)}$ such that $\pi(x_0, x_0') = (y, y')$. Let u be a minimal idempotent such that $(y, y')u = (y, y')$. Then $\pi(x_0, x_0')u = (y, y')$, and (since $\overline{P(X)}$ is closed invariant) $(x_0, x_0')u \in \overline{P(X)}$. ∎

For k a positive integer, let

$$D_k = \{\alpha \in G : gr(\alpha) \subset \overline{P_k(M)}\} = \{\alpha \in G : gr(\alpha) \subset P_{k+1}(M)\}.$$

In particular $D_1 = \{\alpha \in G : gr(\alpha) \subset \overline{P(M)}\} = \{\alpha \in G : gr(\alpha) \subset P_2(M)\}$. Note that the D_k are closed subsets of G. Also, since $\theta(P_k(M)) = P_k(M)$ for $\theta \in G$, it follows easily that D_k is preserved under conjugation by elements of G.

Let D be the closed subgroup of G generated by D_1.

THEOREM 4.2: *Let F be a closed subgroup of G containing D_1. Let (X, T) be a flow with group F. Then (X, T) is a proximal extension of a distal flow.*

Proof: It is sufficient to show that $P(X)$ is closed. For this, in turn, it is sufficient to show that $\overline{P(X)}$ has no non-trivial almost periodic points (by (viii) of Proposition 1.1). Let (x, x') be an almost periodic point of $\overline{P(X)}$ and (if $\pi: M \to X$ is a homomorphism) let (m, m') be an almost periodic point in $\overline{P(M)}$ with $\pi(m, m') = (x, x')$. Then $m' = \alpha(m)$ with $\alpha \in D_1 \subset F$. Hence $\pi\alpha = \pi$, so $x = x'$. ∎

COROLLARY 4.3: (i) *Let (X, T) be a distal minimal flow with group A. Then $D_1 \subset A$.*

(ii) *D is the group of the universal distal minimal flow.*

(iii) *$D_k \subset D$, for $k = 1, 2, \ldots$.*

Proof: (i) Let $\pi: M \to X$ be a homomorphism, and let $x = \pi(m)$ where $mu = m$. If $\alpha \in D_1$, then $\pi\alpha(m) \in \overline{P(x)} = \{x\}$, so $\pi\alpha = \pi$, and $\alpha \in \mathcal{G}(X) = A$.

(ii) By Theorem 4.2, D is the group of a distal minimal flow. If (X, T) is a distal minimal flow with group A, then by (i), $D_1 \subset A$, so $D \subset A$.

(iii) If $\alpha \in D_k$, $gr(\alpha) \subset P_k(M) \subset S_d(M)$, so $\alpha \in D$. ∎

LEMMA 4.4: (i) *If $\alpha \in D_1$, $(id \times \alpha)(P_k) \subset P_{k+1}$ for $k \geq 1$.*

(ii) *$D_1 D_k \subset D_{k+1}$ $(k = 1, 2, \ldots)$.*

(iii) *$D = \overline{\bigcup\{D_1{}^k : k = 1, 2, \ldots\}} = \overline{\bigcup\{D_k : k = 1, 2, \ldots\}}$.*

Proof: (i) The proof is by induction on k. $(id \times \alpha)(P_1) = C((id \times \alpha)(\Delta)) \subset C(\overline{P}) = P_2$.

Suppose $(id \times \alpha)P_{k-1} \subset P_k$. Then

$$(id \times \alpha)P_k = (id \times \alpha)(C(\overline{P_{k-1}})) = C(\overline{(id \times \alpha)(P_{k-1})}) \subset C(\overline{P_k}) = P_{k+1}.$$

(ii) Let $\alpha \in D_1$, $\beta \in D_k$. Then $\underline{(id \times \alpha\beta)(x, x)} = \underline{(id \times \alpha)}(id \times \beta)(x, x) \in (id \times \alpha)(\overline{P}_k) = (id \times \alpha)\overline{C(\overline{P}_{k-1})} = \overline{C((id \times \alpha)(\overline{P}_{k-1}))} \subset \overline{C(\overline{P}_k)} = \overline{P}_{k+1}$.

(iii) It follows easily from (ii) that $D_1^k \subset D_k$, and clearly $\bigcup\{D_1^k : k = 1, 2, \ldots\}$ is a group. Since the closure of a subgroup of G is a group,

$$D = \overline{\bigcup\{D_1^k : k = 1, 2, \ldots\}} \subset \overline{\bigcup\{D_k : k = 1, 2, \ldots\}},$$

which is a subset of D. ∎

Our next result is an alternate characterization of the distal structure relation for a minimal flow. Let R be a symmetric and reflexive relation, and let $\mathcal{E}(R)$ be the equivalence relation generated by R. Thus $\mathcal{E}(R) = \bigcup\{R^n : n = 1, 2, \ldots\}$.

THEOREM 4.5: *Let (X,T) be a minimal flow. Then $S_d(X) = C(\overline{\mathcal{E}(\overline{P})})$.*

Proof: First suppose that X is the universal minimal flow M. It is sufficient to show that $S_d(M) \subset C(\overline{\mathcal{E}(\overline{P})})$, since clearly the opposite inclusion holds.

Let $(x,y) \in S_d(M)$. Then (xu, yu) is an almost periodic point of S_d. It is sufficient to show that $(xu,yu) \in \overline{\mathcal{E}(\overline{P})}$. Now $yu = \delta(xu)$, where $\delta \in D$. Now $\bigcup\{D_1^n : n = 1,2,\ldots\}$ is dense in D, so there are $\delta_i \in D_1^{k_i}$ (for some $k_i > 0$) such that $\delta_i \to \delta$. It follows from the definition of convergence in G that there are $x_i \in M$ such that $(x_i, \delta_i(x_i)) \to (xu, \delta(xu))$. We have $(x_i, \delta_i(x_i)) \in \overline{P}^{k_i}$ so $(xu, yu) = (xu, \delta(xu)) \in \overline{\mathcal{E}(\overline{P})}$.

Now suppose (X,T) is a minimal flow, and let $\pi\colon M \to X$ be a homomorphism. Then $S_d(X) = \pi(S_d(M)) = \pi(C(\overline{\mathcal{E}(\overline{P(M)})})) = C(\pi(\overline{\mathcal{E}(\overline{P(M)})})) \subset C(\overline{\mathcal{E}(\overline{P(X)})})$. ∎

THEOREM 4.6: *Let (X,T) be a minimal flow, and let k be a positive integer. Then $P_k(X) \subset \overline{P(X)}^k \subset P_{k+1}(X)$. Thus if $\delta(X) = k$, then $S_d(X) = \overline{P(X)}^k$. Conversely, if $S_d(X) = \overline{P(X)}^k$ then $\delta(X) = k$ or $k+1$.*

Proof: $P_1 = P$ and $P_2 = C(\overline{P}) = P\overline{P} \subset \overline{P}^2$. Suppose $P_k \subset \overline{P}^k$. Then $P_{k+1} = C(\overline{P_k}) = P\overline{P_k} \subset \overline{P}\overline{P}^k = \overline{P}^{k+1}$.

To prove the second inclusion, first suppose $X = M$, the universal minimal flow, and let $(m, m') \in \overline{P}^k$. Then, if u is a minimal idempotent, $(mu, m'u) \in \overline{P}^k$, so $m'u = \delta(mu)$ where $\delta \in D_1^k \subset D_k$. Then $(mu, m'u) \in \overline{P_k}$ so $(m, m') \in C(\overline{P_k}) = P_{k+1}$. Note that this discussion shows that an almost periodic point of $\overline{P}^k$ is in $\overline{P_k}$.

Now let (X,T) be a minimal flow, and let $\pi\colon M \to X$ be a homomorphism. Let (x,x') be an almost periodic point in $\overline{P(X)}^k$. We show that there is an almost periodic point $(m,m') \in \overline{P(M)}^k$ such that $\pi(m,m') = (x,x')$. This is proved by induction on k. The case $k=1$ is known. Suppose it holds for k and let (x,x') be an almost periodic point of $\overline{P(X)}^{k+1}$. Then $(x,y) \in \overline{P(X)}^k$, $(y,x') \in \overline{P(X)}$ for some y and we may assume that (x,y) and (y,x') are almost periodic. Let $(m, n_0) \in \overline{P(M)}^k$ and $(n,n') \in \overline{P(M)}$ be almost periodic with $\pi(m,n_0) = (x,y)$ and $\pi(n,n') = (y,x')$. We may also suppose that (n,n_0) is almost periodic, so $n_0 = \alpha(n)$ for some $\alpha \in G$. Since $\pi(n) = \pi(n_0)$ we have $\pi\alpha = \pi$. Then $(n_0, \alpha(n')) = (\alpha(n), \alpha(n')) \in \overline{P(M)}$, so $(m, \alpha(n')) \in \overline{P(M)}^{k+1}$ with $\pi(m, \alpha(n')) = (x,x')$.

Suppose now that $(x,x') \in \overline{P(X)}^k$. If u is a minimal idempotent, $(xu, x'u)$ is almost periodic, so $(xu, x'u) = \pi(m,m')$ where (m,m') is an almost periodic point of $\overline{P(M)}^k$. Thus $(m, m') \in \overline{P_k(M)}$, so $(xu, x'u) \in \pi(\overline{P_k(M)}) = \overline{P_k(X)}$. Therefore $(x, x') \in C(\overline{P_k(X)}) = P_{k+1}(X)$. ∎

THEOREM 4.7: *Let (X,T) be a minimal flow. Then $S_d(X) = P_\gamma(X)$, where $\gamma \leq \omega$. Thus the distal order $\delta(X,T)$ of any minimal flow is $\leq \omega$.*

Proof: By Theorem 4.6 for every positive integer k, $\overline{P}^k \subset P_{k+1}$ so $\mathcal{E}(\overline{P}) \subset \bigcup_{k\geq 1} P_k$ and $S_d(X) = C(\mathcal{E}(\overline{P})) \subset C(\overline{\bigcup_{k\geq 1} P_k}) = P_\omega$. ∎

§5. More about the relations P_k

THEOREM 5.1: *Let $\pi: X \to Y$ be a homomorphism of minimal flows. Let γ be a positive integer or ω. Then $\pi(P_\gamma(X)) = P_\gamma(Y)$.*

Proof: It is well known (see also Lemma 2.2) that $\pi(P(X)) = P(Y)$, and therefore $\pi(\overline{P(X)}) = \overline{P(Y)}$. The result for positive integers now follows by induction, using Lemma 2.2. Induction also shows that $\pi(P_\omega)(X) \subset P_\omega(Y)$. If $(y, y') \in \overline{\bigcup_{i<\omega} P_i(Y)})$, then $(y, y') = \lim(y_i, y_i')$ where $(y_i, y_i') \in P_{s_i}(Y) = \pi P_{s_i}(X)$. We may suppose $(x_i, x_i') \to (x, x') \in P_\omega(X)$. Then $(y, y') = \pi(x, x') \in \pi(P_\omega(X))$. An application of Lemma 2.2 finishes the proof. ∎

Let (X,T) be a minimal flow with $\mathcal{G}(X) = A$. Let $\pi: M \to X$ be a homomorphism.

LEMMA 5.2: *Let $g \in G$. Then $\pi(gr(g)) \subset \overline{P_\gamma(X)}$ if and only if $g \in AD_\gamma$.*

Proof: If $g = \alpha\delta$ with $\alpha \in A$ and $\delta \in D_\gamma$, then, if $m \in M$, $\pi(m, g(m)) = \pi(m, \alpha\delta(m)) = \pi(m, \delta(m)) \in \pi(\overline{P_\gamma(M)}) = \overline{P_\gamma(X)}$. Suppose $\pi(gr(g)) \in \overline{P_\gamma(X)}$. Then $\pi(m, g(m)) = \pi(m_0, \delta(m_0))$ where $\delta \in D_\gamma$. We may suppose $mu = m$ and $m_0u = m_0$. Then $m_0 = a(m)$ and $\delta(m_0) = \alpha g(m)$, where $a, \alpha \in A$. It follows that $g = \alpha^{-1}\delta a \in AD_\gamma$. ∎

THEOREM 5.3: *Let $k > 1$ be a positive integer. Then the following are equivalent:*

(i) $S_d(X) = P_k(X)$; i.e., $\delta(X) \leq k$.
(ii) $D \subset AD_{k-1}$.
(iii) AD_{k-1} *is a group.*
(iv) $AD = AD_{k-1}$.
(v) $P_k(X)$ *is an equivalence relation.*

(vi) *$P_k(X)$ is closed.*

Proof: Suppose $D \subset AD_{k-1}$, and let $(x,y) \in S_d(X)$. Then, if u is a minimal idempotent, (xu, yu) is an almost periodic point in $S_d(X)$. Let (m,n) be an almost periodic point in $S_d(M)$ with $\pi(m,n) = (xu, yu)$. Then $n = \delta(m)$ where $\delta \in D$, so $\delta\alpha\delta'$ with $\alpha \in A$ and $\delta' \in D_{k-1}$. Then $(xu, yu) = \pi(m,n) = \pi(m, \delta(m)) = \pi(m, \alpha\delta'(m)) = \pi(m, \delta'(m)) \in \pi(gr(\delta')) \in \pi(\overline{P_{k-1}(M)}) = \overline{P_{k-1}(X)}$. Therefore $(x,y) \in C(\overline{P_{k-1}(X)}) = P_k(X)$.

Suppose $S_d(X) = P_k(X)$, and let $\delta \in D$. Let $m \in M$ with $mu = m$. Then $(m, \delta(m)) \in S_d(X)$, so $(x,y) = \pi(m, \delta(m))$ is an almost periodic point of $S_d(X) = P_k(X)$, so $(x,y) \in \overline{P_{k-1}(X)}$. Then $\pi(gr(\delta)) \in \overline{P_{k-1}(X)}$ and, by Lemma 5.2, $\delta \in AD_{k-1}$.

Hence (i) and (ii) are equivalent. Obviously (i) $\Longrightarrow$ (v) and (vi), and (since $D_1 \subset D_k \subset AD_k \subset AD$, which is a group) the equivalence of (ii), (iii), and (iv) follows easily.

If $P_k(X)$ is closed, then it is a closed capturing set containing Δ, so $P_k(X) = S_d(X)$, and therefore (vi) $\Longrightarrow$ (i).

Finally, we show that (v) $\Longrightarrow$ (ii). Suppose $P_k(X)$ is an equivalence relation. Suppose $\alpha_i \in AD_{k-1}$, $i = 1, 2$; i.e. (Lemma 5.2) $\pi(gr(\alpha_i)) \subset \overline{P_{k-1}(X)} \subset P_k(X)$. Thus we can find $m \in M$ such that $\pi(m, \alpha_1(m))$ and $\pi(\alpha_1(m), \alpha_2\alpha_1(m))$ are in $P_k(X)$. By our assumption also $\pi(m, \alpha_2\alpha_1(m)) \in P_k(X)$; i.e., $\alpha_2\alpha_1 \in AD_{k-1}$. Now Lemma 4.4 implies $D \subset AD_{k-1}$. ∎

THEOREM 5.4: *Let (X,T) and (Y,T) be minimal flows and let $\pi\colon X \to Y$ be a homomorphism. Then $\delta(Y) \le \delta(X)$. If the homomorphism π is proximal, and $\delta(Y) > 0$, then $\delta(Y) = \delta(X)$.*

Proof: Suppose $\delta(X) = \gamma$. Then by Theorems 2.1 and 5.1, $S_d(Y) = \pi(S_d(X)) = \pi(P_\gamma(X)) = P_\gamma(Y)$. Therefore $\delta(Y) \le \gamma = \delta(X)$.

Now suppose π is proximal. If $\delta(Y) = \omega$, then it follows from the first part of this proof that $\delta(X) = \omega$. Suppose $\delta(Y) = k$, a positive integer. We show that $\pi^{-1}(P_k(Y)) = P_k(X)$ and (using Theorem 2.1) this implies the equality of the distal orders. It is known that $\pi^{-1}(P(Y)) = P(X)$, so suppose $k > 1$. Let $\pi(x,x') = (y,y') \in P_k(Y)$. Let u be a minimal idempotent such that $(yu, y'u) \in \overline{P_{k-1}(Y)}$. Then $\pi(xu, x'u) = (y,y')$. Let (x_0, x_0') be an almost periodic point in $\overline{P_{k-1}(X)}$ such that $\pi(x_0, x_0') = (y,y')$. Since π is proximal, there is a unique minimal set in $\pi^{-1}(yu, y'u)T$, so $(xu, x'u) \in \overline{(x_0, x_0')T} \subset \overline{P_{k-1}(X)}$, and $(x,x') \in C(\overline{P_{k-1}(X)}) = P_k(X)$. ∎

§6. A characterization of the equicontinuous structure relation for minimal flows

Next, we characterize the equicontinuous structure relation for minimal flows in terms of the capturing relation.

Let $E_0 = \{\alpha \in G : gr(\alpha) \subset Q\}$ and let $E = \{\alpha \in G : gr(\alpha) \subset S_{eq}(M)\}$. Recall that $D = \{\alpha \in G : gr(\alpha) \subset S_d(M)\}$ is the group of the universal distal minimal flow. It is easy to see that D and E are normal subgroups of G. E_0 is closed under inversion and conjugation by elements of G but in general is not a subgroup of G.

LEMMA 6.1: $E = E_0D$.

Proof: Let $\epsilon \in E$, and let $\pi: M \to M_d$ (where $M_d = M/S_d(M)$, the universal distal minimal flow). If $m \in M$, $(m, \epsilon(m)) \in S_{eq}(M)$, so $\pi(m, \epsilon(m)) \in S_{eq}(M_d)$. Now Q is an equivalence relation in distal minimal flows, so $\pi(m, \epsilon(m)) \in Q(M_d)$. Let $n \in M$ such that $(m, n) \in Q(M)$ and $\pi(m, n) = \pi(m, \epsilon(m))$. We may assume that (m, n) is an almost periodic point, so there is an $h \in E_0$ such that $n = h(m)$. Let $\delta \in G$ such that $\delta(n) = \epsilon(m)$. Now $\pi(\delta(n)) = \pi(\epsilon(m)) = \pi(n)$, so $\pi\delta = \pi$ and therefore $\delta \in D$. Also $\epsilon(m) = \delta(h(m))$, so $\epsilon = \delta h \in DE_0 = E_0D$. ∎

We remark that it can be shown that there is a subgroup G' of G such that $E = G'D$. However, we will not need this stronger result.

The following result, which relates the distal and equicontinuous structure relations, was communicated to us by Bob Ellis.

THEOREM 6.2: *Let (X, T) be a minimal flow. Then $S_{eq}(X) = Q(X)S_d(X)$.*

Proof: It is sufficient to prove the result for $X = M$, the universal minimal flow. The general case will follow from Theorem 2.1. Let $(x, y) \in S_{eq}(M)$. Suppose $xu = x$. Then (x, yu) is an almost periodic point of $S_{eq}(M)$, so $y = \varepsilon(x)$ where $\varepsilon \in E$. By Lemma 6.1, $\varepsilon = \delta\alpha$ where $\delta \in D$ and $\alpha \in E_0$. Then $(\alpha(x), yu) \in S_d(M)$ and $(x, \alpha(x)) \in Q$, so $(x, yu) \in QS_d(M)$. Now $(y, yu) \in P \subset S_d(M)$ so $(x, y) \in QS_d(M)$. ∎

Let (X, T) be a minimal flow, and let $Q^\#(X) = Q^\#$ be the smallest closed capturing set containing $Q(X)$. Let $\pi: M \to X$ be a homomorphism, and let $K = \{(a, b) \in M \times M : \pi(a, h(b)) \in Q^\# \text{ for all } h \in E_0\}$.

LEMMA 6.3: *K is a closed capturing set containing $S_d(M)$.*

Proof: Since $Q^\#$ is closed and capturing, the same holds for K. If $a \in M$ and $h \in E_0$, then $(a, h(a)) \in Q(M)$ so $\pi(a, h(a)) \in Q(X) \subset Q^\#$, and $(a, a) \in K$.

Therefore $\Delta \subset K$, and since K is closed and capturing, it follows from Theorem 3.3 that $S_d(M) \subset K$. ∎

THEOREM 6.4: *Let (X,T) be a minimal flow. Then $Q^{\#} = S_{eq}$.*

Proof: Since $S_{eq}(X)$ is closed, capturing, and contains Q, $Q^{\#} \subset S_{eq}(X)$. Let (x,y) be an almost periodic point of $S_{eq}(X)$ and let $(a,b) \in S_{eq}(M)$ be almost periodic with $\pi(a,b) = (x,y)$. Then $b = \varepsilon(a)$ where $\varepsilon \in E$. Since $E = E_0 D$, we have $\varepsilon = h\delta$ with $h \in E_0$ and $\delta \in D$. Now $(a, \delta(a)) \in S_d(M) \subset K$. By definition of K, $(x,y) = \pi(a,b) = \pi(a,\varepsilon(a)) = \pi(a, h\delta(a)) \in Q^{\#}$.

Therefore, all almost periodic points of $S_{eq}(X)$ are in $Q^{\#}$. Since $Q^{\#}$ is capturing, it follows that $S_{eq}(X) \subset Q^{\#}$. ∎

COROLLARY 6.5: *Let (X,T) be a minimal flow. Then the following are equivalent:*

(i) *Q is an equivalence relation.*

(ii) $C(Q) = Q$.

(iii) $PQ = Q$.

Proof: This follows from Lemma 1.2 and Theorem 6.4. ∎

§7. The equicontinuity order

For a flow (X,T) let $Q_1 = Q(X)$, $Q_2 = C(Q_1)$ and inductively, if γ is an ordinal number and Q_i has been defined for $i < \gamma$, define $Q_\gamma = C(\overline{\bigcup_{i<\gamma} Q_i})$. (Therefore $Q_{\gamma+1} = C(\overline{Q_\gamma})$.)

The **equicontinuity order** of (X,T) (denoted $\epsilon = \epsilon(X,T)$) is defined to be the least ordinal number γ for which the equicontinuity structure relation $S_{eq}(X) = Q_\gamma(X)$. ($\epsilon(X,T) = 1$ if and only if $Q(X,T)$ is an equivalence relation.)

For k a positive integer, let $E_k = \{\alpha \in G : gr(\alpha) \subset \overline{Q_k(M)}\}$. In particular $E_0 = \{\alpha \in G : gr(\alpha) \subset Q(M)\}$. Note that the E_k are closed subsets of G. Also, since $\theta(Q_k(M)) = Q_k(M)$ for $\theta \in G$, it follows easily that E_k is preserved under conjugation by elements of G. Let $E = \mathcal{G}(M/S_{eq})$, the Ellis group of the universal equicontinuous minimal flow; then we have $E = E_{\epsilon(M)}$.

LEMMA 7.1: $E = E_0 E_\omega$.

Proof: Since clearly $P_k \subset Q_k$ for every $k \le \omega$, we have $D_k \subset E_k$, hence (by Lemma 5.4) $D \subset E_\omega$. By Lemma 6.1, this implies $E = E_0 E_\omega$. ∎

Question: Does $E = E_\omega$?

Example 7.2: Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let a and b be the self-homeomorphisms of X defined by

$$a(t) = t + \alpha \pmod 1; \quad b(t) = \left(t - \frac{[4t]}{4}\right)^2 + \frac{[4t]}{4}, \quad 0 \le t < 1,$$

and let Γ be the group generated by a and b. McMahon introduced the flow (X, Γ) as an example of a minimal and weakly mixing flow where the relation $Q(X)$ is not an equivalence relation ([M]). It is easy to see that in this flow, $P = \{(x, x') : |x - x'| < 1/4\}$ and $\overline{P} = Q = \{(x, x') : |x - x'| \le 1/4\}$. Moreover, we have

$$Q_2 = P_2 = C(Q) = C(\overline{P}) = \{(x, x') : |x - x'| < 1/2\},$$

whence $\overline{C}(Q) = \{(x, x') : |x - x'| \le 1/2\} = X \times X = S_{eq} = S_d$. Thus McMahon's example has distal and equicontinuity order 3.

If, for an integer $n > 2$, we let

$$a(t) = t + \alpha \pmod 1; \quad b(t) = \left(t - \frac{[nt]}{n}\right)^2 + \frac{[nt]}{n}, \quad 0 \le t < 1,$$

we obtain a flow (X, Γ) with distal and equicontinuity order k, where k is the least integer such that $k/n > \frac{1}{2}$; i.e., $k = [n/2] + 1$ (i.e., $Q_{k-1} \subsetneq Q_k = X \times X$).

In the next section we will have to construct more sophisticated examples in order to get minimal cascades (i.e., $\mathbb{Z}$-flows) with distal order exactly k, $k = 0, 1, 2, \ldots$, and $k = \omega$.

We conclude this section with the following observations.

Another approach to the equicontinuity order is to define it via the "prolongation". One defines for a subset $A \subset X$ of a flow (X, T) its prolongation as the set

$$\Pi(A) = \{x : \exists \text{ nets } \{x_i\} \text{ in } X, \{t_i\} \text{ in } T \text{ such that } x_i \to x,\ x_i t_i \to z \in A\}$$

The proof of the following lemma is immediate.

LEMMA 7.3: (i) *If A is closed, then $\Pi(A)$ is closed.*

(ii) $\Pi(\Delta) = Q$.

(iii) *Let $\pi\colon X \to Y$ be a flow homomorphism, and let $K \subset X$. Then $\pi(\Pi(K)) \subset \Pi(\pi(K))$. If φ is an automorphism of (X, T), $\varphi(\Pi(K)) = \Pi(\varphi(K))$.*

(iv) $C(A) \subset \Pi(A)$.

Note that, for any flow, $Q = \Pi(\Delta)$. Now set $Q'_0 = \Delta$, $Q'_1 = Q = \Pi(Q'_0)$, and for any ordinal γ, $Q'_\gamma = \Pi\left(\overline{\cup_{\alpha<\gamma} Q'_\alpha}\right)$ (so $Q'_{\gamma+1} = \Pi(Q'_\gamma)$).

It is easy to see that $\Pi(S_{eq}) \subset S_{eq}$, so $Q'_\gamma \subset S_{eq}$. Also, it follows from (iv) of Lemma 7.3 that $Q_\gamma \subset Q'_\gamma$. Therefore, there is an ordinal number $\gamma \le \epsilon(X)$ for which $Q'_\gamma = S_{eq}(X)$. The alternative definition of the equicontinuity order is to define it as $\epsilon'(X)$, the smallest ordinal for which $Q'_\gamma = S_{eq}(X)$. This is the same ordinal for which the Q'_γ stabilize ($Q'_{\gamma+1} = Q'_\gamma$). Clearly $\epsilon'(X) \le \epsilon(X)$.

Corresponding to Theorem 4.7, we have:

THEOREM 7.4: *Let (X,T) be a minimal flow. Then $S_{eq}(X) = Q'_\gamma$ where $\gamma \le \omega$. Thus $\epsilon'(X) \le \omega$.*

We omit the proof, which is a close paraphrase of the proof in section 4 of the result that the distal order is $\le \omega$. One replaces the D_k with $F_k = \{\alpha \in G : gr(\alpha) \subset Q'_k\}$.

§8. Examples

In this section we use the letter τ to denote the generator of every $\mathbb{Z}$ flow.

PROPOSITION 8.1: *For every positive integer $k \ge 0$, as well as for $k = \omega$, there exists a minimal metric flow with distal order k.*

Proof: Our examples are obtained via a slight modification of an example of L. Shapiro, [S]. This is a minimal flow for which proximal is an equivalence relation but is not closed.

We will briefly recall the construction in [S], keeping the same notations. Start with the circle $K = \{k \in \mathbb{C} : |k| = 1\}$ and an element $\lambda \in K$ which is not a root of unity. Thus the flow (K, τ), where $\tau k = \lambda k$, is an irrational rotation. Next choose an infinite sequence $k_1, k_2, \ldots \in K$ such that the orbits $\mathcal{O}(k_i) = \{\lambda^n k_i : n \in \mathbb{Z}\}$ are mutually disjoint. Let $E = \bigcup_i \mathcal{O}(k_i)$ and assume further that $1 \notin E$. The space X consists of the points $\{k^\pm : k \in K\}$ where $k^+ = k^-$ iff $k \notin E$. The map ϕ: $X \to K$ is given by $\phi(k^\pm) = k$. The topology on X is defined as follows. We let the "intervals" $[m^-, k^+] := \phi^{-1}(m,k) \cup \{m^-, k^+\}$, where (m,k) is an arc on K traversed counterclockwise from m to k, form a subbase for the topology on X. Clearly each subset $[m^-, k^+]$ is both open and closed, so that X is a zero-dimensional space. It is also easy to see that it is compact and that the map ϕ is continuous.

Define τ: $X \to X$ by $\tau(k^\pm) = (\lambda k)^\pm$. We can easily check that τ is a homeomorphism of X, so that (X, τ) is a minimal flow, and that ϕ: $(X,\tau) \to (K,\tau)$ becomes a flow homomorphism, in fact an almost 1-1 extension. Moreover, we have

$$P(X,\tau) = A(X,\tau) = \{(x,x') : \exists k \in K \ \{x,x'\} = \{k^-,k^+\}\},$$

where $A(X,\tau)$ denotes the set of doubly asymptotic pairs in $X \times X$, i.e., those pairs (x,x') with

$$\lim_{|n|\to\infty} d(\tau^n x, \tau^n x') = 0,$$

for some compatible metric d.

The next stage of the construction is to form a group extension $Y = X \times G$ of X where G is a compact abelian group. This is done by means of a "cocycle", i.e., a continuous $f\colon X \to G$. Once f is given the flow on Y is defined by

$$\tau(x,g) = (\tau x, gf(x)).$$

The following auxiliary functions are then defined. First the cocycle $f_n(x)$ is given by

$$f_n(x) = \begin{cases} \prod_{i=0}^{n-1} f(\tau^i x) & \text{for } n \geq 0, \\ \prod_{i=n}^{-1} f(\tau^i x) & \text{for } n < 0, \end{cases}$$

so that $\tau^n(x,g) = (\tau^n x, gf_n(x))$. For $k \in K$ set $\rho_i(k) = f(\tau^i k^+)f(\tau^i k^-)^{-1}$; then

$$\sigma_n(k) = \begin{cases} \prod_{i=0}^{n-1} \rho_i(k) & \text{for } n \geq 0, \\ \prod_{i=n}^{-1} \rho_i(k)^{-1} & \text{for } n < 0, \end{cases}$$

and $\xi(k) = \lim_{|n|\to\infty} \sigma_n(k)$, when it exists. We recall the following Lemma (Lemmas 2 and 3 and Corollary 1 of [S]).

Lemma 8.2:

(1) *$\xi(k)$ exists iff for each $g \in G$ there is a unique $g_1 \in G$ such that (k^+, g) is proximal to (k^-, gg_1). If $\xi(k)$ exists, then $\xi(k) = g_1$ and $\xi(\lambda^m k)$ exists for all $m \in \mathbb{Z}$.*

(2) *The proximal relation $P(Y)$ is an equivalence relation iff $\xi(k)$ exists for every $k \in K$. When it is an equivalence relation*

$$P(Y) = \Delta \cup \{((k^+,g),(k^-,g\xi(k))) : k \in K, g \in G\}$$
$$\cup \{((k^-,g),(k^+,g\xi(k)^{-1})) : k \in K, g \in G\}.$$

(3) *$P(Y)$ is closed iff $\xi(k)$ exists for every $k \in K$ (i.e., $P(Y)$ is an equivalence relation) and $\lim_\nu \xi(k_\nu) = g$ for a sequence k_ν of distinct elements in K implies $g = e$, the identity element of G.*

We now take $G = K$ and refer to [S] for the details of the construction of the cocycle $f\colon X \to K$. In fact, the only change needed is in the definition of the sequence ϵ_m. We let $\epsilon_1 = \exp(i\theta_1)$ for an arbitrary number $\theta_1 \in (0,\pi]$, and then for $m \geq 1$ we let $\epsilon_m = \exp(i\theta_1/m)$. We also assume, as we clearly may, that the sequences $\{p(n,j) : |j| \leq n\}$ satisfy

$$\min\{|p(n,j) - p(n,k)| : |j|, |k| \leq n,\ j \neq k\} \to \infty.$$

We now have a family of flows (Y_{θ_1}, τ) with parameter $\theta_1 \in (0, \pi]$, the **Shapiro flows**. As we will presently see they are all minimal flows.

We next show that the distal order of Shapiro's flow (Y_{θ_1}, τ) is the least n for which $n\theta_1 \geq \pi$.

By definition we have $P_0 = \Delta$ and $P_1 = C(P_0) = P(Y)$, the proximal relation on Y. We prove our proposition by establishing by induction the following chain of claims.

CLAIM 1:

$$\begin{aligned} P_1 = C(P_0) = P_0 &\cup \{((k^+, g), (k^-, g\xi(k))) : k \in K, g \in G\} \\ &\cup \{((k^-, g), (k^+, g\xi(k)^{-1})) : k \in K, g \in G\}. \end{aligned}$$

Proof: We recall that it is shown in [S] that in the Shapiro flows $\xi(k)$ exists for every $k \in K$. Now use Lemma 8.2 (2). ∎

CLAIM 1′: $\overline{P}_1 = P_1 \cup \{((x, g), (x, g\exp(i\theta))) : x \in X, g \in G, \theta \in [-\theta_1, \theta_1]\}$.

Proof: In the list of observations which lead to the conclusion that $P(Y)$ is a non-closed equivalence relation ([S], p. 524) we note (g) $\xi(k_n) = \epsilon_1$ for all n, and (i) $\xi(k) = 1$ for $k \notin E$. It is also easy to see that, for fixed $n > 0$, the set

$$\{\xi(\tau^\ell k_n) : |\ell| \leq \max\{|p(n,j)|, |j| \leq n\}\}$$

is equal to the set $\{\epsilon_n^j : |j| \leq n\}$. It follows that the range of ξ is $\{\epsilon_n^j : n = 1, 2, \ldots, |j| \leq n\}$. Moreover, our assumption on the sequences $\{p(n,j) : |j| \leq n\}$ implies that $\xi(\tau^\ell(k_n))$ is constant for long sequences of integers. Hence if N is a positive integer, then for n sufficiently large each ϵ_n^j is equal to $\xi(\tau^\ell(k_n))$ for (at least) N consecutive integers ℓ. Our claim now follows from the identification of $P(Y)$ in Claim 1 and the observation that for a minimal cascade (X, τ) with X metric, given $\varepsilon > 0$ there is a positive integer N such that, for every $x \in X$, the set $\{x, \tau(x), \ldots, \tau^N(x)\}$ is ε dense in X. ∎

CLAIM 2:

$$\begin{aligned} P_2 &= C(\overline{P}_1) \\ &= \overline{P}_1 \\ &\cup \{((k^+, g), (k^-, g\exp(i\theta)\xi(k))) : k \in K, g \in G, \theta \in [-\theta_1, \theta_1]\} \\ &\cup \{((k^-, g), (k^+, g\exp(i\theta)\xi(k)^{-1})) : k \in K, g \in G, \theta \in [-\theta_1, \theta_1]\}. \end{aligned}$$

Proof: This follows from Proposition 1.1 (vi) and Claim 1. ∎

CLAIM 2′: $\overline{P}_2 = P_2 \cup \{((x,g),(x,g\exp(i\theta))) : x \in X, g \in G, \theta \in [-2\theta_1, 2\theta_1]\}$.

Proof: Same as that of Claim 1′. ∎

In the same way we prove for every positive integer $n \geq 1$ two claims:

CLAIM n:

$$\begin{aligned} P_n &= C(\overline{P}_{n-1}) \\ &= \overline{P}_{n-1} \\ &\cup\{((k^+,g),(k^-,g\exp(i\theta)\xi(k))) \colon k \in K, g \in G, \theta \in [-(n-1)\theta_1,(n-1)\theta_1]\} \\ &\cup\{((k^-,g),(k^+,g\exp(i\theta)\xi(k)^{-1})) \colon k \in K, g \in G, \theta \in [-(n-1)\theta_1,(n-1)\theta_1]\} \end{aligned}$$

and

CLAIM n': $\overline{P}_n = P_n \cup \{((x,g),(x,g\exp(i\theta))) : x \in X, g \in G, \theta \in [-n\theta_1, n\theta_1]\}$.

It now follows that for every $n \geq 1$ with $n\theta_1 < \pi$, P_n properly contains P_{n-1} and we deduce that the Shapiro flow (Y_{θ_1}, τ) has distal order k, where k is the least positive integer with $k\theta_1 \geq \pi$. This completes the proof for the case $k \geq 1$ an integer.

For $k = \omega$ we take any sequence Y_j, $j = 1, 2, \ldots$, of minimal flows where the distal order of Y_j is $\geq j$. Let Y_∞ be any minimal joining of the countable family Y_j; then clearly the distal order of Y_∞ is ω. In fact it is easy to check that $S_d(Y_\infty)$ is a subset of $\overline{\bigcup_n P_n(Y_\infty)}$.

Finally, a minimal flow is distal iff it has distal order 0. The proof of Proposition 8.1 is now complete. ∎

COROLLARY 8.3: *Shapiro's flow (Y, τ) is minimal.*

Proof: Let $L \subset Y$ be a fixed minimal subset of Y. Denoting the action of $k \in K$ on Y by R_k: $(x, k') \mapsto (x, k'k)$, we see that

$$K_0 = \{k \in K : R_k L \cap L \neq \emptyset\} = \{k \in K : R_k L = L\}$$

is a closed subgroup of K and that $Y = \bigcup_{k \in K} R_k L$. It follows that the relation

$$S = \{(y, y') \colon y \text{ and } y' \text{ are in the same minimal subset}\}$$

is a closed equivalence relation. It is now easy to see that $P_1 = P(Y) \subset S$ and then, by induction, that for every n also $P_n \subset S$. Since we established that for some n, $P_n = S_{eq}(Y) = R_\pi$, where $\pi\colon Y \to K$ is the homomorphism $\pi(x,k) = \phi(x)$ and

$$R_\pi = \{(y,y') : \pi(y) = \pi(y')\},$$

it now follows that $R_\pi \subset S$. In particular, $K_0 = K$ and $Y = L$ is minimal. ∎

Remark: As we have already observed, a minimal flow is distal if and only if it has distal order $k = 0$. Since in a minimal flow the proximal relation has the property that it is an equivalence relation when it is closed, it follows that a minimal flow has distal order $k = 1$ iff its proximal relation is closed. Since $\overline{P} = X \times X = S_d$ in a weakly mixing flow X with T abelian, the distal order of such a flow is 2. It is not hard to see that the Morse flow also has distal order $k = 2$.

References

[A] J. Auslander, *Minimal Flows and their Extensions*, Notas de Matemática, 1988.

[E] R. Ellis, *Lectures on Topological Dynamics*, W. A. Benjamin, Inc., New York, 1969.

[EG] R. Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups*, Transactions of the American Mathematical Society **94** (1960), 272–281.

[M] D. C. McMahon, *Weak mixing and a note on the structure theorem for minimal transformation groups*, Illinois Journal of Mathematics **20** (1976), 186–197.

[S] L. Shapiro, *Proximality in minimal transformation groups*, Proceedings of the American Mathematical Society **26** (1970), 521–525.

[V] J. de Vries, *Elements of Topological Dynamics*, Kluwer Academic Publishers, Dordrecht, 1993.